Ex: The constant function $r=5$ is a circle.

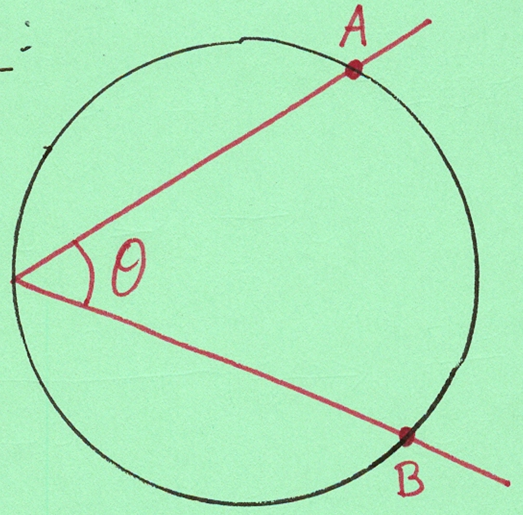
Since $f(\theta)=5$, $f'(\theta)=0$, and since its graph is a circle, $\gamma=\frac{\pi}{2}$. Thus

$$0=f'(\theta)=f(\theta)\cdot\tan\left(\gamma-\frac{\pi}{2}\right)=5\cdot\tan(0)=0 \checkmark$$

Ex: Verify $f'(\theta)=f(\theta)\tan\left(\gamma-\frac{\pi}{2}\right)$ for $f(\theta)=\cos\theta$.

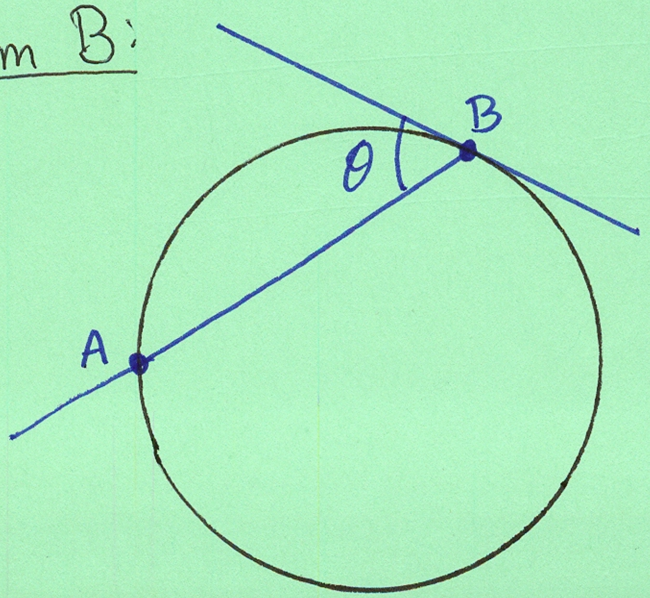
Sol: Recall some geometry facts

Theorem A:



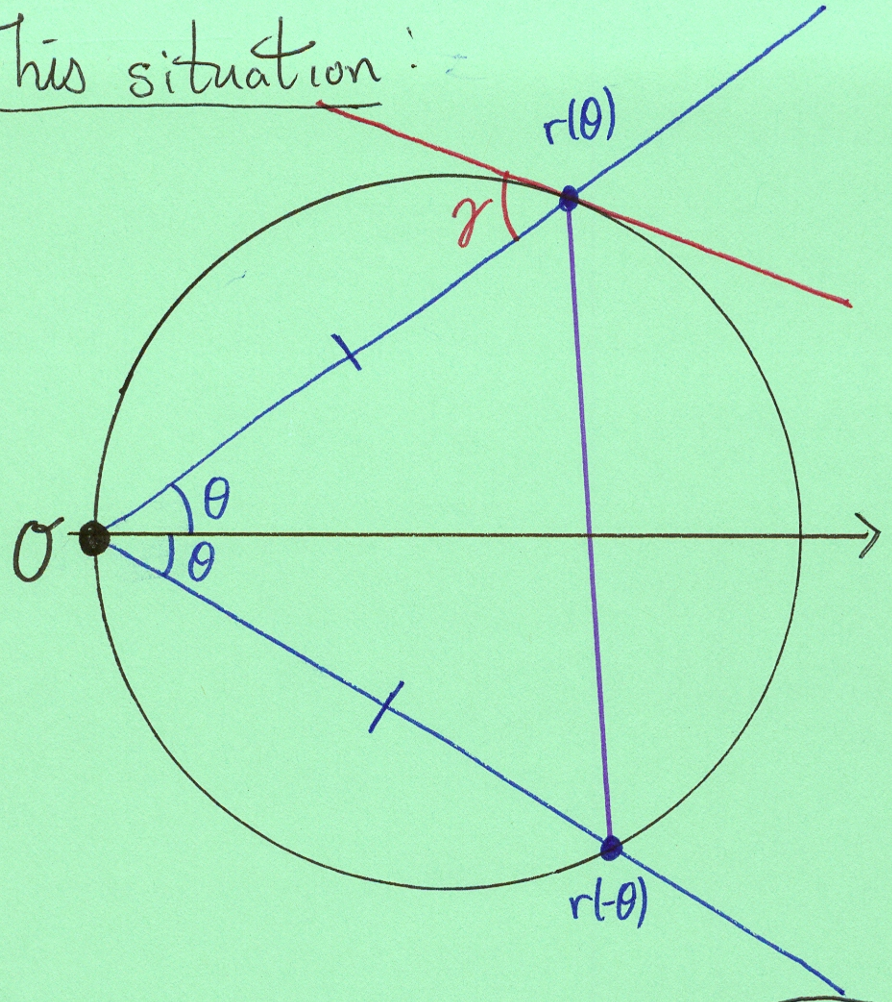
$$\theta=\frac{1}{2}m\widehat{AB}$$

Theorem B:



$$\theta=\frac{1}{2}m\widehat{AB}$$

This situation:



① Theorem A $\Rightarrow 2\theta = \frac{1}{2} m\,\overset{\frown}{r(-\theta)r(\theta)}$

② $m\,\overset{\frown}{\mathcal{O}r(\theta)} = m\,\overset{\frown}{\mathcal{O}r(-\theta)}$

③ Theorem B $\Rightarrow \gamma = \frac{1}{2} m\,\overset{\frown}{\mathcal{O}r(\theta)}$

④ $2\pi = m\,\overset{\frown}{\mathcal{O}r(\theta)} + m\,\overset{\frown}{r(-\theta)r(\theta)} + m\,\overset{\frown}{r(-\theta)\mathcal{O}} = 2m\,\overset{\frown}{\mathcal{O}r(\theta)} + m\,\overset{\frown}{r(-\theta)r(\theta)}$

$\Rightarrow 2\pi = 4\gamma + 4\theta \Rightarrow \theta + \gamma = \frac{\pi}{2} \Rightarrow \gamma = \frac{\pi}{2} - \theta$

So, $\tan(\gamma - \frac{\pi}{2}) = \tan(-\theta) = -\tan\theta$

$f(\theta) = \cos\theta \Rightarrow f'(\theta) = -\sin\theta$

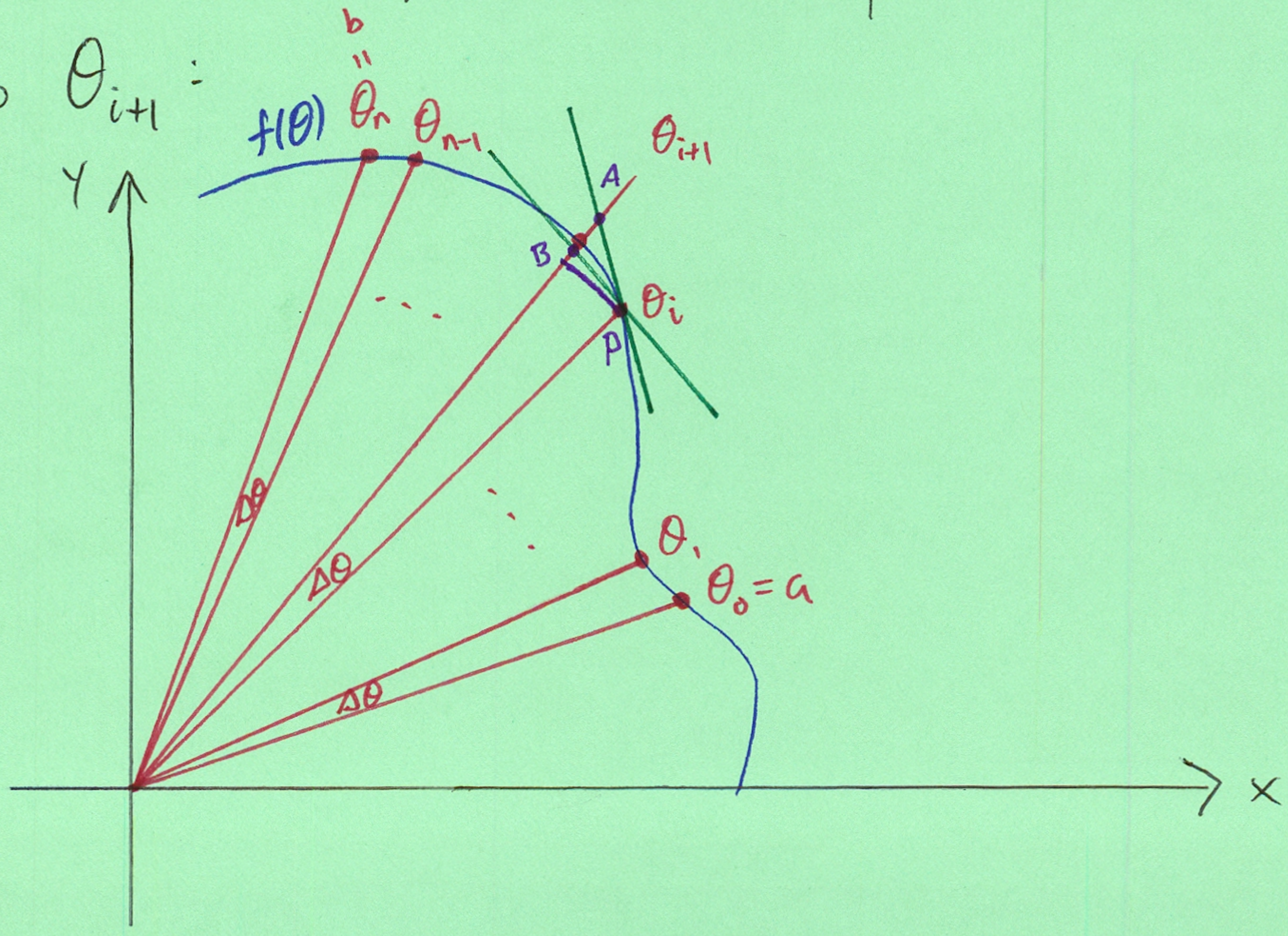
$f(\theta)\tan(\gamma - \frac{\pi}{2}) = \cos\theta(-\tan\theta) = -\sin\theta$ ✓

Lengths of Polar Curves

Let $a \le \theta \le b$ and assume $f(\theta)$ is differentiable on this interval. Let n be a very large number and partition $[a, b]$ into n pieces:

$$a = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_i < \theta_{i+1} < \cdots < \theta_{n-1} < \theta_n = b.$$

$\Delta\theta = \theta_{i+1} - \theta_i$ and let L_i be the length of the piece of $f(\theta)$ between θ_i and θ_{i+1}. Our analysis of the graph of $f(\theta)$ when studying $f'(\theta)$ will serve us again here: Let's focus on the portion from θ_i to θ_{i+1}:



We need to approximate L_i. Because n is very large, $\Delta\theta$ will be very small. L_i will be well approximated by AP. For very small $\Delta\theta$, ΔABP is essentially a right triangle. Thus

$$AP \approx \sqrt{AB^2 + BP^2}$$

If Δs_i is the arc length of the circle at P between θ_i & θ_{i+1}, then for small $\Delta\theta$

$$BP \approx \Delta s_i = f(\theta_i)\,\Delta\theta$$

Also, since $\Delta\theta$ is very small

$$f'(\theta_i) \approx \frac{f(\theta_i + \Delta\theta) - f(\theta_i)}{\Delta\theta}$$

so

$$AB \approx f(\theta_i + \Delta\theta) - f(\theta_i) \approx f'(\theta_i)\,\Delta\theta$$

Thus: $L_i \approx AP \approx \sqrt{AB^2 + BP^2} \approx \sqrt{[f'(\theta_i)\Delta\theta]^2 + [f(\theta_i)\Delta\theta]^2}$

$$= \sqrt{[f(\theta_i)]^2 + [f'(\theta_i)]^2}\,\Delta\theta$$

Finally,

$$L = \lim_{n\to\infty} \sum_{i=0}^{n-1} L_i = \lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{[f(\theta_i)]^2 + [f'(\theta_i)]^2}\,\Delta\theta$$

$$= \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$$

Ex: Find the length of the polar curve

$r = \cos\theta$.

Sol: To complete one loop of $r=\cos\theta$, we need $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$.

$f(\theta) = \cos\theta$, $f'(\theta) = -\sin\theta$

$$\Rightarrow L = \int_{-\pi/2}^{\pi/2} \sqrt{\cos^2\theta + (-\sin\theta)^2}\, d\theta = \int_{-\pi/2}^{\pi/2} \sqrt{\cos^2\theta + \sin^2\theta}\, d\theta = \int_{-\pi/2}^{\pi/2} \sqrt{1}\, d\theta$$

$$= \theta \Big|_{-\pi/2}^{\pi/2} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}$$

Does this make sense?

Areas in Polar Coordinates

Let $f(\theta)$ be continuous on $a \le \theta \le b$. We would like to find the area swept out by the line segment from $(0,\theta)$ to $(f(\theta),\theta)$ as θ goes from a to b.